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Thesis

'μ-Free Strategies: Analysis of the Global Minimum Variance Portfolio (GMVP) and the Most Diversified Portfolio (MDP)'

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SUMMARY

Introduction

In this Thesis we introduce 2 asset allocation strategies: the Global Minimum Variance Portfolio and Most Diversified Portfolio. We present the theoretical concepts behind these μ-free strategies therefore seeing the mathematic formulas and applications of these. There is also an implementation of the two strategies with real data in which we make a comparison between these allocation optimisations to analyse their performance, diversification, and impact on the portfolio's standard deviation (through risk decomposition). The writer's interest in asset allocation began thanks to Markowitz's Mean-Variance Model leading thus to go into depth on portfolio construction, focusing in this case on 2 of the many risk-based strategies. The purpose of this thesis is to further explore and enhance knowledge within portfolio construction.

This study begins with a first section in which we present one of the three phases of the asset allocation: the strategic asset allocation. Asset classes are the focal point of this phase in which we determine our allocation universe based on other inputs that we will analyse later. After having implemented these elements, we introduce Markowitz's Mean-Variance Model, which is has been published in 1952. On one hand, it is globally known for being a solid base of today's Modern Portfolio Theory (MPT). On the other hand, other than being revolutionary in the asset allocation sector, this model has major downsides which directly affect its performance therefore raising doubts about its effectiveness.

In the second section, we present the so called μ-free strategies in which from its name we can deduce that the focus is not anymore on the estimated return dimension. Since another input is prioritized, we need to integrate our study with other quantitative formulas to fully understand the impact this input has on portfolios. These are presented as risk decompositions in a portfolio. Now that we have all the metrics and fundamentals, we finally propose the 2 approaches we mentioned earlier. The first approach is the Global Minimum Variance approach (GMVP) characterized by having an optimisation like Markowitz's Mean-Variance Model that we have mentioned earlier. The second and final approach is the Most Diversified approach which is a very recent one (2008) and it takes inspiration from the Sharpe Ratio (SR) invented in 1966.

This thesis's last chapter focuses on the implementation of the two techniques we've seen in the previous chapter. We apply them to real data from 2001 to 2023. Other than putting into practice the GMVP and MD approaches, we also perform a comparative analysis between them, emphasizing their performance, diversification and especially to which investor do they aim. We also focus on the risk decomposition methods we introduced earlier since these measures help us dive deep into the impact of any variation of the risk dimension in the portfolio.

CHAPTER 1

The methodologies and issues of Markowitz's Model: The Mean-Variance Optimisation

1.1 Strategic Asset Allocation SAA

Asset Management can be defined as a practice of increasing total wealth over time by acquiring, maintaining, and trading investments that have the potential to grow in value. Asset allocation is a branch of asset management and can be described as an investment strategy in which asset classes are chosen from a universe. Its aim is to develop a portfolio using the chosen asset classes to create a balanced diversified combination that has the best trade-off between risk and estimated return. It is important to specify that it is done considering the investor's financial goals. An investor must be open to the concept of financial portfolios. Why? Simply because it is smart to put our own money to work for us. As stated previously, financial portfolios are made of asset classes which are real instruments that must be generated from a healthy process. Once a portfolio is created it must be monitored to have a hold on its fluctuations therefore if needed modify it and rebalance the allocations easily.

The investment process can be defined in 3 phases: the first being the Strategic Asset Allocation SAA; the second being the selection of actual products to implement and finally the last phase being the Tactical Asset Allocation TAA. In the Strategic Asset Allocation, we have a set of portfolio propositions for the investor based on his long-term objectives that have a certain risk and return profile. These objectives are usually valued by the advisor that questions the client's risk tolerance, ideal expected returns, knowledge in finance which is known as the MiFID Complaint. Certainly, one of the prerequisites of our study is having a rational investor that sees the estimated return as a positive element and the standard deviation as a negative one. The investor would want a higher estimated return for a given risk and a lower risk for a given expected return. However, we should emphasize that every investor perceives the risk metric differently from other investors. To summarise it: the investor should approach investments by weighting the risks and returns while maintaining a long-term perspective.

Now that we've introduced some general concepts, we proceed with defining portfolio's components in the following paragraphs.

1.2 Asset Classes: Expected Returns and volatilities

A portfolio is a range of investments that can be held by an individual or an organisation. Each portfolio is characterized by asset classes: we can define each asset class as a group of investments that have in common some characteristics. To make it more understandable for the reader we can imagine your grocery basket in which you have your apples, your lettuce, and your strawberries. Each group of fruit represents an asset class. Each strawberry differs from the other, however all of them together differ from other fruits. Now let's apply the same concept into finance: imagine you have the same basket but rather than vegetables and fruits you have a few investments in European bonds, other investments in gold and the remaining invested in the tech industry.

As you may have noticed all of these are grouped differently since they all have different characteristics, however among each group every single constituent has something in common with the other constituents; in our case the keyword 'European', 'Gold' and 'Tech'. For example, investor X's portfolio can have 40% invested in NVIDIA since the tech industry is in its bull season during this time, another 30% invested in Pfizer and the remaining 30% invested in Lockheed Martin seen that there are some geopolitical concerns around the world. Now that we made the concept of asset classes clear, we should consider analysing the elements that constitute these.

The first element is the estimated return which is also known as the sample mean. There are two different types of means that can be used to have an in depth understanding of the asset class's performance. The first one is the arithmetic mean, and it is used to estimate the average or most common value in a set of data. The investor that uses this metric sees each return as a single and separate return: in statistics each variable is IID which stands for Independent and Identically Distributed. IID refers to each variable having the same distribution as the other variables and is independent from the others and is also equiprobable from the others. We use the arithmetic mean to statistically estimate the most plausible value of the return since our goal is to quantify the estimated return for our future investments. Having a sample of an asset class's monthly returns, we can estimate the expected return of this asset class. Let's presume investor X has invested in asset class A 100$ thus having the following monthly fluctuations.

100$ invested in A throughout a year	Percentage Change
101.98	$= \frac{101.98-100}{100}$ = 1.98%
102.58	$= \frac{102.58-101.98}{101.98}$ = 0.588%
98.78	$= \frac{98.78-102.58}{102.58}$ = -3.704%

101.23	$=\frac{101.23-98.78}{98.78}$ = 2.48%
105.87	$=\frac{105.87-101.23}{101.23}$ = 4.584%
103.38	$=\frac{103.38-105.87}{105.87}$ = -2.352%
105.63	$=\frac{105.63-103.38}{103.38}$ = 2.176%
106.79	$=\frac{106.79-105.63}{105.63}$ = 1.098%
106.45	$=\frac{106.45-106.79}{106.79}$ = -0.318%
104.96	$=\frac{104.96-106.45}{106.45}$ = -1.4%
107.63	$=\frac{107.63-104.96}{104.96}$ = 2.544%
109.66	$=\frac{109.66-107.63}{107.63}$ = 1.886%

Now that we've seen how to calculate monthly percentage for an investment made in asset class A, we must analyse what investor X has seen in his investment throughout the year. We must introduce the Geometric Average Return metric (also known as Annualised Return) to see ex-post the average growth of his/her investment. For this purpose, we introduce the following formula which calculates the growth rate using the final investment amount 'M' with the initial capital 'C' invested for a 't' period:

$$R=\left(\frac{M}{C}\right)^{1/t}-1$$

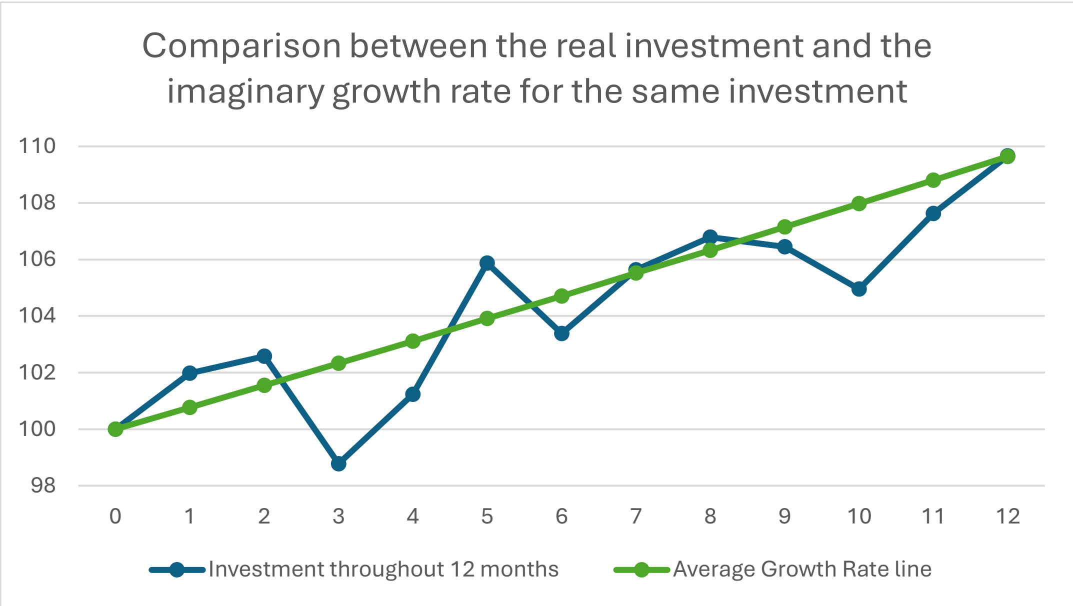
We shall now proceed to implement this concept within the context of our example:

$$R=\left(\frac{109.66}{100}\right)^{1/12}-1\cong 0.771\%$$

As stated earlier, this quantitative formula determines the geometric growth rate of our investment. It is as if investor X has seen his/her investment grow with a rate of 0.77% each month. In our case, it is a monthly percentage growth rate and to make it annual we simply use this formula:

$$R_{Annual} = (1+0.771\%)^{12} - 1 = 9.655\%$$

However, as we may know, each month the investment had a different percentage change and none of them were 0.77%. It is important to specify it since we might fall for the mistake of saying that the investment itself has grown each month about 0.77% whereas each month it has been registered a different percentage, and we have even observed some negative fluctuations throughout the investment period. We can imagine this average growth rate as an increasing curve to see, ex-post, how the investment itself grew. To make it more understandable for the reader, we introduce the next graphic based on the previous example.



After analysing the geometric average return, we now turn our attention to the other metric we introduced earlier: the arithmetic mean. This metric is simply the sum of returns divided by the total number of returns. The arithmetic mean is also known as estimated value in statistics. This quantitative method has as an output the most plausible value of the investment's return. It means that the average monthly return is this value. However, it could never be equal to the real value of the return that we calculate ex-post using the geometric return method. This problem is caused by another important input that we are introducing later: the volatility also called standard deviation. It is important to state that the geometric return will always be lower than the Estimated Return due to this other input.

$$\bar{R} = \frac{\sum_{i=1}^{n} r_i}{n}$$

We shall now proceed to implement this concept within the context of our example using these following variables:

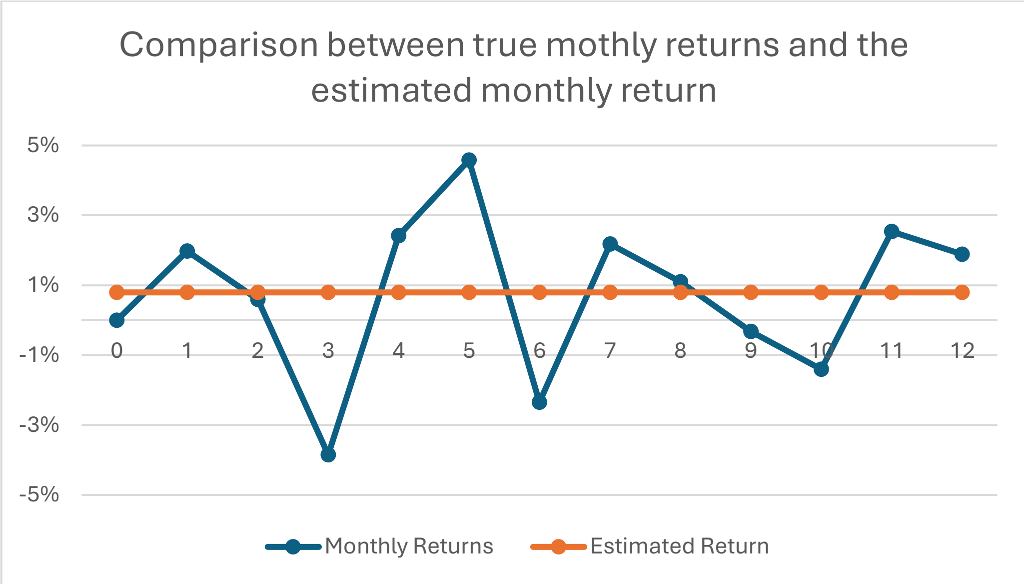
Months	Returns
1	1,98%
2	0,59%
3	-3,70%
4	2,48%
5	4,58%
6	-2,35%
7	2,18%
8	1,10%
9	-0,32%
10	-1,40%
11	2,54%
12	1,89%

Our average return would be the sum of these 12 monthly returns divided by the number of returns itself which is in this case 12.

$$\bar{R} = \frac{\sum_{i=1}^{12} r_i}{12} = 0.797\%$$

We simply multiply the average return to 12 to get the annual return for this investment: in this case it is about 9.562%.

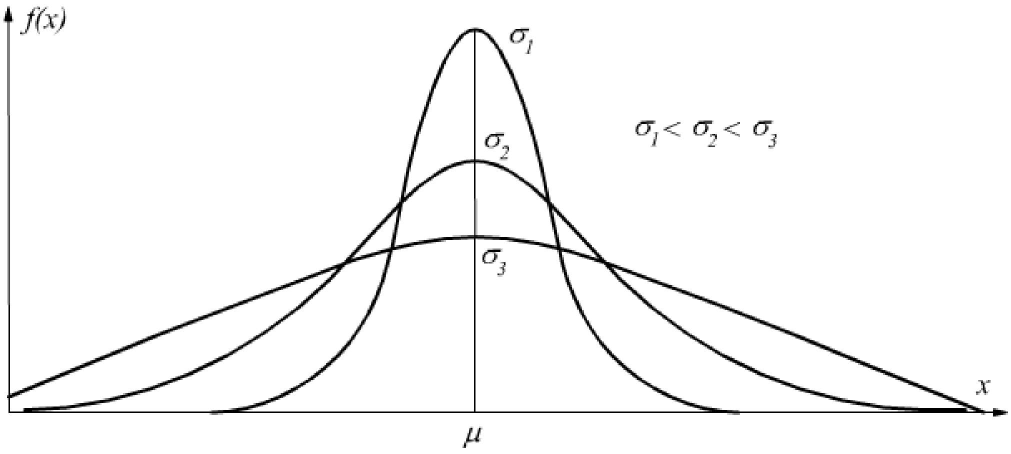
As you may have already noticed, the arithmetic mean is higher than the Annualised Return as expected. The arithmetic mean is a metric that is used more by future investors who must evaluate whether to invest in this asset class or not. This metric is more optimistic since it has a higher value, however the investor must consider the volatility measure since it is the reason for the gap generated between the Expected Return and the geometric average return. In our example it isn't a significant gap, however in some cases it can be a substantial difference caused by a high standard deviation.



Now that we've introduced the Annualised Return and the Estimated Return, we must introduce the Standard Deviation, also known as the volatility. As seen earlier, this metric is the reason for which there is a discrepancy between the Expected Return and the geometric average return. The Standard Deviation is the risk component of an investment, and we can define it as an exposure to uncertainty. The way each monthly return happened to be different from the estimated return is caused by investor X's exposure to uncertainty. It does not make investor X's fault rather than knowing that every single investment has a risk component. Standard deviation is the square root of the Variance. It is a measure of dispersion of a return from the average mean which is the estimated value we've seen earlier. This also means that the lower the variance σ^2 is, the closer monthly returns are to the estimated return. We introduce the following formula for Standard Deviation:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{T}(R_t - \bar{R})^2}{T-1}}$$

To make it more understandable, we introduce the following distributions:



https://www.edutecnica.it/calcolo/normale/5.png

In the previous picture, we have three distributions with three different standard deviations. It is noticeable that the thinner this symmetric distribution is, the closer we are to μ which is for us the exact return we get from our investment. In conclusion, the lower the standard deviation is, the smaller the dispersion will be between each return and the estimated return. We continue with the previous example therefore we calculate the volatility for investing in asset class A:

$$\sigma \cong 2.357\%$$

To have the Annual Volatility, we introduce the following formula:

$$\sigma_{Annualised} = \sigma * \sqrt{12}$$

The yearly standard deviation for investing in asset class A is: $\sigma \cong 2.378\% * \sqrt{12} \cong 8.165\%$. This also means that most of investor X's monthly returns are in this following range: $(9.562\% - 8.165\%; 9.562\% + 8.165\%)$ which equals to $(1.397\%; 17.727\%)$. It is important to note that it does not make all returns fit into this range, instead, it is possible that some end up outside of it.

To conclude this paragraph, Returns and Volatilities are two important measures to consider when investing. The first measure indicates the gains (or losses) we receive from the investment while the second measure is somehow the negative aspect that comes from an investment. We can approximate the distribution of these returns to a Gaussian distribution since these two measures both need to be considered, therefore we have $r \sim N(\mathrm{R}, \sigma)$. Having made these concepts clear, it should be natural for us to say that an investor would want a higher positive return for a lower volatility. We will introduce in the next paragraph a model which has been the solid base of Modern Portfolio Theory that focuses on these two measures while considering the investor's expectations.

1.3 The Mean-Variance Model

The Mean-Variance Optimisation Model is a mathematical framework introduced by the economist Harry Markowitz in 1952. The purpose of this model is to minimize the volatility of a portfolio given a certain level of return. The Mean-Variance Optimisation is at the base of modern portfolio construction. It consists in resolving the weight distribution of asset classes, chosen from a universe, to achieve the most efficient trade-off between risk (σ) and return ($\overline{R}$) according to the investor's preferences. This Model's inputs are the following: estimated returns, standard deviations, and correlations between asset classes.

We've seen in the previous paragraphs how to estimate the return for an investment made in an asset class: we simply use the mean estimator based on the monthly percentage changes. After estimating the return, we calculate the standard deviation for the asset class in which we have seen in the formula the estimated value for the return. Now we can introduce the concept of correlations between asset classes. We can imagine that we must make a soup and when we did grocery shopping, we only bought 10 carrots. If we add or remove a few carrots the outcome will always be something that tastes only like carrots. Instead, if we buy all the ingredients needed for that soup, we will have a beautiful combination of flavours that is well balanced and that gives us the taste of a soup. Now let's apply this same concept into our case: in our portfolio, if we add or remove an investment made in the same asset class, the outcome will always be the same since any major change is going to impact in the same way these investments we made. If we decide to invest in different asset classes that have different characteristics, it will lead to a balanced portfolio where, if faced changes, it will balance itself thanks to the different responses different asset classes have.

A correlation is a measure we use to express these relationships between asset classes to understand whether they react the same way or the opposite way faced some changes. We can express this relationship between asset classes through a statistical densification function in which this densification can be approximated to a straight line. Mostly if the angular coefficient is positive, meaning the straight line is in the first and third quadrant of the Cartesian plane, then both asset classes are harmoniously having similar trend. Instead, if the approximated straight line has a negative angular coefficient, then the function is in the second and forth quadrant. We can say that both asset classes have opposite trends which is a behaviour we want for balancing our portfolio.

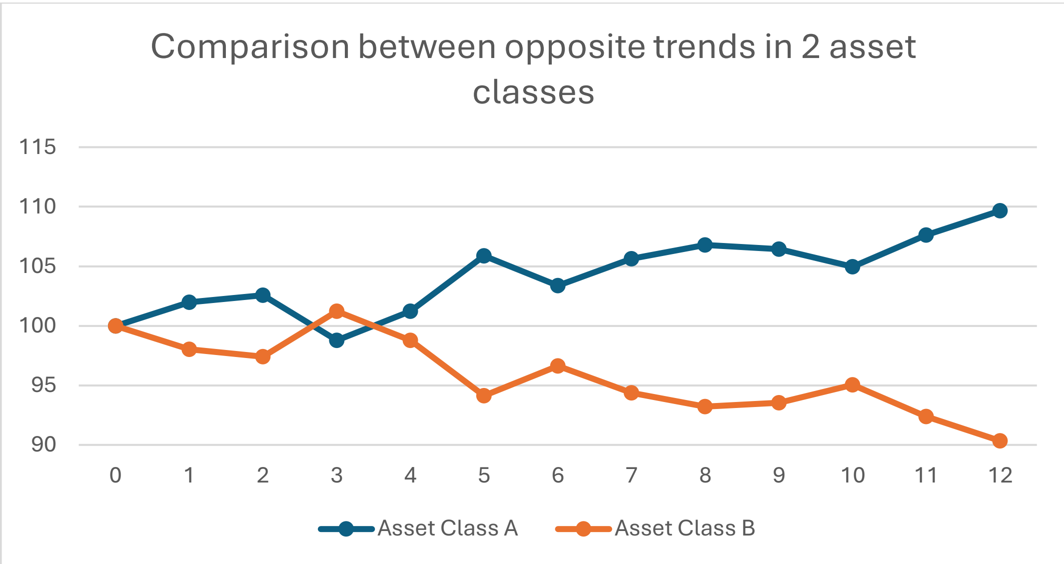
We introduce the following formula for the correlation:

$$CORR_{A;B} = \frac{COV_{A;B}}{\sigma_A * \sigma_B}$$

We also introduce the covariance formula since it is needed to calculate the correlation among asset classes:

$$COV_{A;B} = \frac{\sum_{i=1}^{T}(R_t^A - \bar{R})(R_t^B - \bar{R})}{T-1}$$

We know that standard deviations are always ≥ 0 therefore what influences the correlation's sign is the covariance's sign. In the event of COV > 0 happening, it implies that both markets have had similar trends. Instead, if COV < 0, it would indicate that both asset classes had opposite trends which is an aspect we would want in our portfolio. With the covariance formula we're able to know about the positive or negative relation between two asset classes, however we're not able to know the intensity of this relation. In this case we use the correlation which goes from $\left[-1;1\right]$. The closer we are to -1, the more these asset classes have opposite trends till reaching -1 which is the most inverse relationship between 2 markets.

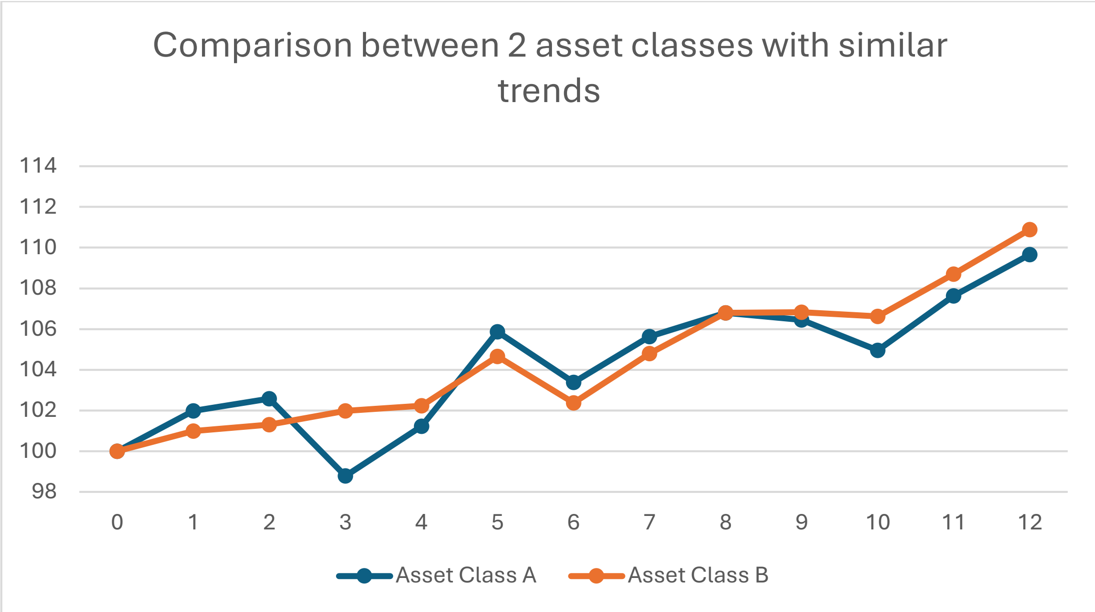


As we can see from the previous graph, we have asset class A (that we've seen in the previous chapters) being compared to asset class B. We will proceed with calculating the estimated returns, covariance, and correlation between these two investments to prove what we have stated earlier.

$$COV_{A;B} = \frac{\sum_{i=1}^{12}(R_t^A - 0.797\%) * (R_t^B + 0.814\%)}{11} = -5.887$$

$$CORR_{A;B} = \frac{-5.887}{2.357\% * 2.501\%} = -0.999 \cong -1$$

We proceed with the same example with similar trends. We introduce the following line graph with 2 similar trends:



As we said previously, when COV>0 it will lead to a positive correlation, meaning both markets have similar trends. We propose the following COV and CORR between these 2 asset classes:

$$COV_{A;B} = \frac{\sum_{i=1}^{12} (R_t^A - 0.797\%) * (R_t^B - 0.873\%)}{11} = 2.187$$

$$CORR_{A;B} = \frac{2.187}{2.357\% * 1.341\%} = 0.692$$

It is now proven how similar trends in asset classes lead to a direct relationship thus having a positive correlation, in this case approximately 1. It can't be CORR = 1 since the two-line graphs do not overlap. It was necessary to introduce the correlation metric since a portfolio is influenced drastically due to the fact of it being one of the inputs. A portfolio that has asset classes that have negative correlations tend to have a less nervous trend which happen to be very helpful emotionally to the investor. On the contrary, a portfolio that has asset classes related positively by the correlation will lead to a nervous trend. The portfolio's risk tends to decrease with the decreasing of the correlation among asset classes.

After introducing the main inputs of this model (estimated returns, standard deviations and correlations/covariances), we finally begin with portfolio construction starting from the portfolio's return.

Let's assume a portfolio A has *n* asset classes, *n* standard deviations and *n* weights. We can define weights as a percentage holding of asset class i. The portfolio's annualised return is calculated through the following formula:

$$R_p = \bar{r}_1 w_1 + \bar{r}_2 w_2 + \bar{r}_3 w_3 .... + \bar{r}_n w_n$$

$$R_p = \sum_{i=1}^{n} (\bar{r}_i \, w_i)$$

In matrix algebra we would have a μ vector (1, N) which is the estimated returns vector as well as W vector which is the weights vector (N, 1):

$$\mu_p = \mu w'$$

Or we can see it this way around:

$$\mu_p = \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 & \dots & \mu_n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \dots \\ w_n \end{pmatrix}$$

Let's introduce the following example to make it more understandable: let's presume investor X has invested in asset class A and asset class B respectively 40% and 60%. The first verification we must make is that the sum of all weights is equal to 1 which is our case in this example. We have the following returns for both asset classes:

Months A	Returns A
1	1,98%
2	0,59%
3	-3,70%
4	2,48%
5	4,58%
6	-2,35%
7	2,18%
8	1,10%
9	-0,32%
10	-1,40%
11	2,54%
12	1,89%

Months B	Returns B
1	-1,98%
2	0,59%
3	-3,85%
4	2,42%
5	4,58%
6	-2,35%
7	2,18%
8	1,10%
9	-0,32%
10	1,40%
11	2,98%
12	2,57%

Respectively we have their annual returns:

$$R_A = 6.82\% \qquad R_B = 9.32\%$$

Following the previous formula, Investor X's portfolio has this return:

$$R_p = 6.82\% * 40\% + 9.32\% * 60\% \cong 8.32\%$$

Calculating a portfolio's return is not a complex task, therefore we introduce as well how to calculate the portfolio's volatility also known as standard deviation in statistics. We cannot apply the same method, meaning we cannot use the weighted average on the asset classes' weights. With this same method, we find the highest volatility that our portfolio could reach, therefore it is not our case since a risk adverse investor would opt for a lower volatility than reaching the highest. This portfolio having the highest volatility lacks diversification: we must know by now that combining asset classes in an efficient way is effective to lower the standard deviation, therefore if we happen to reach the highest volatility, it means that diversification was missing in this portfolio.

We must introduce this diversification formula which summarizes these concepts:

$$\frac{\sum_{i=1}^{n} \sigma_i w_i - \sigma_{REAL}}{\sum_{i=1}^{n} \sigma_i w_i} = x\%$$

With this formula, we're able to quantify the percentage of risk we avoided by combining asset classes. The higher the percentage is, the higher the benefit from diversification will be and the lower the portfolio's volatility is going to be. Now, we introduce with the following formula how to determine the portfolio's volatility:

$$\sigma_p = \sqrt{\sum_{i=1}^{n} (\sigma_i w_i)^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}} (w_i w_j \sigma_i \sigma_j CORR_{ij})}$$

In matrix algebra, we have the following formula:

$$\sigma_p = \sqrt{w'\Sigma w}$$

Or we can see the formula this way around: we calculate the variance and then we apply the square root to find the standard deviation.

$$\sigma^2{}_p = \begin{pmatrix} w_1 & w_2 & \dots & w_n \end{pmatrix} \begin{pmatrix} COV_{11} & \cdots & COV_{1n} \\ \vdots & \ddots & \vdots \\ COV_{n1} & \cdots & COV_{nn} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix}$$

After seeing some relevant elements for the framework, we must focus on the optimization itself, in which Markowitz used firstly an objective function and, secondly, constraints to set some limits to get a certain output. The objective is to identify the portfolio's weights considering a given expected return (target return) and a minimized portfolio variance or standard deviation. The output of this framework would be a Mean-Variance efficient portfolio that represents the best trade-off between risk and return. The objective function consists in minimizing the standard deviation through the only inputs that can be manipulated which are the weights.

$$F(w) = Min\ \sigma_p = Min \sqrt{\sum_{i=1}^{n} (\sigma_i w_i)^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}} (w_i w_j \sigma_i \sigma_j CORR_{ij})}$$

This function alone is insufficient, reason why, there are three constraints necessary to achieve the minimum volatility for a given return. The first constraint is the financial constraint which helps the algorithm reach an expected output which is a specific return we'd like to reach.

We represent this constraint through the following formula:

$$\sum_{i=1}^{n} \overline{R}_i w_i = \overline{R}_p^{\,*}$$

The second constraint is the budget constraint, also known as full investment constraint, in which the sum of all weights must be 1. This means that the portfolio must reach the maximum investor's budget which is 100% of his budget. We introduce this formula for the budget constraint:

$$\sum_{i=1}^{n} w_i = 1$$

The third and last constraint is the long-only constraint, therefore the no short selling constraint. This constraint helps us avoid short selling which we can define it as a speculative game that some investors practice. It consists in selling an investment when we expect it to downturn. During these downturn periods, the investor would buy bonds, stocks at a lower price to sell them later at a higher price. To avoid this, we simply impose that the asset classes' weights are higher or equal to 0.

$$w_i \geq 0$$

In conclusion, Markowitz's framework can be summarized with the following system:

$$\begin{cases} Min \sqrt{\sum_{i=1}^{n}(\sigma_i w_i)^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}} (w_i w_j \sigma_i \sigma_j CORR_{ij})} \\ \sum_{i=1}^{n} \overline{R}_i w_i = \overline{R}_p^{\,*} \\ \sum_{i=1}^{n} w_i = 1 \\ w_i \geq 0 \end{cases}$$

We introduce the following percentage changes that are also called monthly returns to which we will apply the Mean-Variance optimisation:

Years	Returns A.C. 1	Returns A.C. 2	Returns A.C. 3	Returns A.C. 4
0				
1	5,89%	7,89%	-2,25%	2,20%
2	3,67%	5,42%	9,98%	-9,55%
3	2,60%	5,41%	4,16%	-5,05%
4	4,67%	3,90%	-2,98%	3,81%
5	1,85%	-2,16%	6,02%	21,35%
6	3,27%	4,10%	4,16%	6,39%
7	-0,79%	1,49%	2,57%	5,02%
8	0,83%	1,57%	3,08%	-2,30%
9	1,48%	-2,16%	5,13%	9,81%
10	0,71%	4,39%	-2,99%	3,56%
11	0,86%	2,30%	3,91%	3,78%
12	0,09%	1,40%	1,24%	2,18%

Our investment universe has these returns and volatilities for each asset class:

	Estimated Return	Standard Deviation
Asset Class 1	2,10%	1,98%
Asset Class 2	2,80%	3,01%
Asset Class 3	2,67%	3,90%
Asset Class 4	3,43%	7,69%

As we may already know from the objective function, it is needed the covariance matrix and the correlation matrix therefore we introduce these matrixes as well:

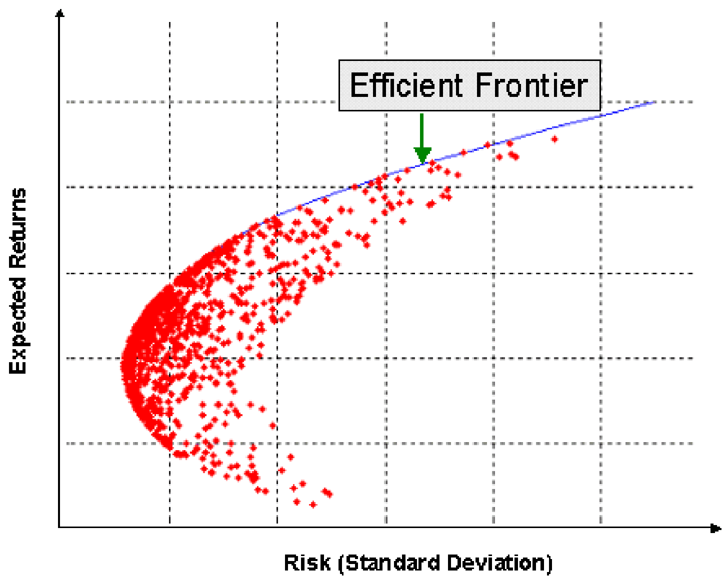
COV X/Y	Asset Class 1	Asset Class 2	Asset class 3	Asset Class 4
Asset Class 1	**0,000390559**	0,000363055	-0,000114367	-0,000261119
Asset Class 2	0,000363055	**0,000905681**	-0,000391391	-0,001562492
Asset Class 3	-0,000114367	-0,000391391	**0,001518809**	-0,00022248
Asset Class 4	-0,000261119	-0,001562492	-0,00022248	**0,005906055**

CORR X/Y	Asset Class 1	Asset Class 2	Asset class 3	Asset Class 4
Asset Class 1	**1**	0,610438556	-0,148493089	-0,171927908
Asset Class 2	0,610438556	**1**	-0,333711436	-0,675587038
Asset Class 3	-0,148493089	-0,333711436	**1**	-0,074283378
Asset Class 4	-0,171927908	-0,675587038	-0,074283378	**1**

We proceed with implementing the optimization framework to obtain the weights needed to create a portfolio that has a return target set by us with its minimum standard deviation. We can notice that the budget constraint is respected since the sum of all weights equals to 1. The long-only constraint is also maintained in this resolution since all weights' percentage are positive. Another check we could do is verifying that the portfolio's risk is lower compared comparison to the weighted average risk (diversification ratio) which in our example it is respected.

	Annual Return	Annual Std Dev	Weights
Asset Class 1	2,10%	1,98%	12%
Asset Class 2	2,80%	3,01%	47%
Asset Class 3	2,67%	3,90%	24%
Asset Class 4	3,43%	7,69%	16%
Portfolio	2,78%	1,14%	100%
Target Return	2,78%		
Weighted average risk	3,84%		
Diversification benefit	70,20%		

With this framework, we managed to create a portfolio having an expected target return of 2.78% and a risk percentage of 1,14. The weighted average risk, which is the highest risk level that a portfolio can reach, is also 70.20% higher than the portfolio's risk. This can also be seen as the portfolio's risk being 70.20% lower than the weighted average risk thus saving the investor from an unwanted risky percentage. After this application for this framework, we must introduce the model's complications that will lead us to look for alternative frameworks. The portfolio that comes from the Mean-Variance framework is the most efficient portfolio, therefore we find it in the so called 'Efficient Frontier'. In this curve we have all optimized portfolios for their given returns, the lowest risk percentages and, vice versa, for the lowest risk percentages we have their highest returns. On the left side of this curve, there cannot be any portfolio since its logic it's quite unrealistic: a portfolio with a 9% yearly return combined with a 1% risk does not exist (sadly) therefore on the left side there are no portfolios. However, on the right side of the curve, we find many portfolios having not the best trade-off between risk and return. On the far-left side of this curve, there is a portfolio having the lowest risk which is called Global Minimum Variance Portfolio: we will be analysing later this portfolio with its optimization. We present for the reader the Efficient Frontier by Markowitz:



https://marketxls.com/wp-content/uploads/2020/08/1817586f9fc789d9db7e25ee703c4357-1.png

1.4 The Model's complications: Estimation Errors in the Expected Returns

Markowitz's Mean-Variance optimization has been revolutionary for the Asset Management field: it gave a rational solution to portfolio construction's problems. As we've seen previously, this framework is mainly based on expected returns, standard deviations and covariances or correlations. We have also seen that these inputs haven't been given, therefore we had to estimate them.

The Mean-Variance optimizer uses the variables we give as an input as if they were the true parameters of. This suggests that this framework is neglecting the uncertainty that comes from these inputs. Many professors have pointed out significant problems in this model. In this field, it is quite known that estimation errors in the risk measure are not errors we should worry about since their impact is minimal. These estimation errors create a sort of a gap between the estimated value of the parameter and its true or real value. We can summarize the model's defaults in the next paragraphs.

Firstly, the Mean-Variance Portfolio is known for lacking diversification which happens to be in contrast with Markowitz's idea of maximizing diversification. We have seen the benefits of maximising diversification such as lowering the portfolio's risk. These efficient allocations we created through the framework often exclude some asset classes that are in the investment universe, therefore giving a significant weight percentage to some others. In the last example we made, we can notice that all 4 asset classes were included respectively with the following percentages: 12%; 47%; 24%; 16%. This also means that we managed to include the whole investment universe, however, if we happen to change the return target, we can notice percentage weights fluctuate and at some point, attribute 0% to some asset classes.

	Annual Return	Annual Std Dev	Weights
Asset Class 1	2,10%	1,98%	0%
Asset Class 2	2,80%	3,01%	27%
Asset Class 3	2,67%	3,90%	0%
Asset Class 4	3,43%	7,69%	73%
Portfolio	3,26%	5,07%	100%
Target Return	3,26%		
Weighted average risk	6,41%		
Diversification benefit	20,86%		

In this example, we set a return target of 3.26% which is our Investor's preferred return. As we can see there is a relevant difference between the weights of our previous example and the ones from this example. It is noticeable how 2 asset classes have been left out of the portfolio. Another important aspect is the significant weight given to the fourth asset class which proves exactly what we stated earlier. The concentration of the weight in asset class 4 did not happen by coincidence, instead we set a high target return of 3.26% and to reach this return, the algorithm had to shift most of the weights to the asset class that has the highest return which is in our case asset class 4. Other than weighting asset class 4, we have some diversification benefit from investing in asset class 2 as well. Now let's imagine if the investor had wanted a target return of 3.96%. Obviously, this portfolio is not able to reach 3.96% since the highest return is 3.43% due to this fact, this portfolio is impossible to create. This optimizer tends to heavily weight those asset classes that show high estimated returns compared to low variances having negative correlations. For this reason, these asset classes most likely suffer from high estimation errors. As said by Michaud (1989): '*The intuitive character of many optimized portfolios can be traced to the fact that Mean-Variance optimizers are, in a fundamental sense, estimation-errors maximisers*'.

This framework's second issue is the instability of its portfolios. These are very sensitive in terms of changes in the estimated parameters. Chopra (1993) has proven that a slight change of estimated value from estimators have led to a different optimized mean-variance portfolio. Similar asset classes when they're facing small perturbations or fluctuations may completely alter the distribution of the portfolio's weights. This happens because there might be an alternation between the dominant and the dominated asset class. This instability caused by these fluctuations has a practical impact: in fact, changes on expected returns tend to impact drastically the weights of a mean-variance portfolio.

	Annual Return	Annual Std Dev	Weights
Asset Class 1	2,10%	1,98%	13%
Asset Class 2	2,80%	3,01%	47%
Asset Class 3	2,69%	3,90%	24%
Asset Class 4	3,43%	7,69%	16%
Portfolio	2,78%	1,15%	100%
Target Return	2,78%		
Weighted average risk	3,83%		
Diversification benefit	70,05%		

In the previous table, we slightly changed the estimated return of asset class 3 from 2.67% to 2.69% and we can already notice a change in asset class's 1 weights with a change of +1% in comparison to just 0.02% change in the estimated return of asset class 3.

Another issue of this optimizer is that it does not recognize the non-uniqueness of optimal portfolios. Michaud (1989) stated that this optimizer has an output a unique optimal portfolio for a given level of risk, however this statement is not correct. We have an optimal portfolio created by estimated values that have estimation errors, therefore just as the estimated values do not coincide with the real, true value of the parameters, also this optimal portfolio does not coincide with the real portfolio with 0 estimation errors. We stated these to make it understandable that for a given level of risk, this portfolio isn't the real and exact optimal portfolio since there is the issue of estimation errors. For this reason, an optimal portfolio can be unique for a given level of risk only if these inputs are without statistical estimation errors.

The last issue of this model is the poor out-of-sample performance. We based our estimation on a sample period; therefore, the misspecification of the input parameters cannot be ignored. Sample means are subject to large fluctuations, and they happen to have a poor predictive power. It is also known that estimation risks in optimized portfolios come from estimated returns since these are higher than covariance/correlation errors. For this reason, errors in estimated returns are more costly than the others since they're 10 times as important as errors in variances, which are twice as important as errors in covariances. To make it more understandable, our framework is based on a sample from which the number of its components can vary. We also should know that the higher the number of components is, the closer we are to the real average value of the population. If we have few returns in the sample, then we will have broad, very generic, and imprecise estimated values which create optimal portfolios that are too generic having high estimation errors. However, if we have a very detailed sample with as much returns as there are, then the estimated values will be very approximately close to the real values, and with this we are able to create much more precise portfolios therefore we're able to create a much more accurate efficient frontier.

In conclusion, we could say that this framework is very rational and of great quality, however, the inputs that are inserted into this optimizer are not in the same level as the optimizer. This model proceeds with resolving the system considering these inputs as if they are the true parameters, therefore they're considered as if they don't have any estimation errors. As we have seen in the previous paragraphs, estimation errors are significative in estimated returns from which these contribute to the creation of a portfolio full of estimation errors. In the next chapter, we are introducing new quantitative methods that overtake the estimation errors' problem. These methods are the so-called mean-free strategies.

CHAPTER 2

Analysis of Portfolio Strategies: GMVP and MDP

2.1 Introduction to μ-Free Strategies

As we have seen from the previous chapter, the Mean-Variance Optimization lacks precision. However, it is not caused by the optimisation framework itself, rather than the quality of the inputs we insert in this model. The inputs with lowest quality, also meaning with the highest estimation errors, are the estimated returns. It has been proven that these are higher than the covariance/correlation errors and risk related errors. There have been some methods being developed to minimize the issue of estimation errors. These methods are known as the Heuristic method and the Bayesian method. Both methods maintain the Markowitz's framework therefore they rely on the trade-off between the mean and the standard deviation.

However, it also means that both methods rely on the same inputs and for this reason the estimation issue repeats itself. We are introducing in the next paragraphs the Risk-Based strategies that do not rely on the Mean-Variance framework. These strategies, as suggested by their name, do not include in their inputs the expected returns that, as we have seen, happen to be problematic. Other than being known as risk-based strategies, they are also named µ-Free Strategies. If the estimated return dimension is being removed, the remaining significant input is the risk dimension. In fact, this dimension is being put as the core element of these strategies in which they focus on either diversifying or minimizing portfolio's risk. They are also known as low-volatility portfolio construction methods seen how these approaches focus on decreasing the portfolio's standard deviation. Another important aspect to consider is the portfolio choice that differ between MVO and risk-based approaches.

We have seen in the mean-variance framework how to optimize the quadratic function including the target constraint as well as other two constraints. We have also proven in the previous chapter how the portfolio changes when we insert different target returns. This also means that we have many portfolios to choose from seen how we can change the target return that changes by consequence the portfolio's setting. In the low-volatility approaches, the focus in only on the risk metric. This implies that there isn't any target return that we could manipulate. Instead, the framework finds the solution with no target constraint. These strategies recommend one and only asset allocation solution. We will make this concept clearer in the next paragraphs.

2.2 Risk Decomposition in a Portfolio

In this paragraph, we introduce the risk decomposition metrics since these are necessary to comprehend the impact each asset class i has on its portfolio. These metrics focus mainly on the risk dimension, and as we have seen the volatility becomes the focal point of our study. To attribute the total risk of a portfolio to each individual asset class, it is required the use of risk decomposition methods that we are analysing in these paragraphs.

It is suggested to start from the Euler's theorem to understand how an asset class contributes to the overall portfolio risk. According to Euler's principle, if RM is a generic risk measure that can be expressed as a continuously differentiable function of weights vector, and it is homogeneous of degree 1 in the sense that the following equation holds for all $\lambda>0$, then:

$$RM(\lambda w) = \lambda RM(w)$$

Increasing or decreasing the scale of the portfolio increases or decreases the risk measure in the same magnitude. Our risk measure becomes the following formula:

$$RM(w) = \sum_{i=1}^{n} w_i \frac{dRM}{dw_i}$$

The addends in this equation are called Euler contributions that, in our case, are risk contributors. These risk contributors satisfy a full allocation property. Now it is possible for us to identify the two fundamental tools that are needed for the risk decomposition metrics that we are seeing after this short introduction. These tools are the following: the marginal risk contribution (MR_i) and the total risk contribution (TRC_i).

The marginal risk contribution (MR_i) of each asset class is calculated as the first derivative of the selected portfolio risk measure with respect to its weight w_i. This metric tells us the amount of variation in the risk measure (RM) when there is an infinitesimal change in the asset class's weight (W). We introduce the formula for this first tool:

$$MR_i = \frac{dRM}{dw_i}$$

The total risk contribution of each asset class is calculated as the product of the allocation to asset class i with its marginal risk. With the MR metric, we can identify the variation in the RM (general risk measure) caused by one and only unit of the weight dimension of asset class i, however, with the TRC metric, we calculate the entire/total variation in the RM measure therefore, for this reason, we multiply MR with the total weight of the asset class (total units). We introduce the formula for this second fundamental tool:

$$TRC = w_i \frac{dRM}{dw_i}$$

We also introduce another important metric we use to calculate the percentage of risk impact each asset class has in the total risk measure RM. We simply divide the total risk contribution (TRC) to the risk measure (RM), following its formula:

$$PTRC = \frac{TRC_i}{RM} = w_i \frac{dRM}{dw_i} \frac{1}{RM}$$

Now that we introduced these metrics, we can associate RM, which was a general risk measure, to our risk dimension which is the standard deviation. We proceed with implementing the concepts we have seen previously within our traditional measure of portfolio risk.

In our case, the risk measure equals to the portfolio's volatility therefore:

$$RM = \sigma_p = \sqrt{\sum_{i=1}^{n} (\sigma_i w_i)^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}} (w_i w_j \sigma_i \sigma_j CORR_{ij})}$$

The marginal risk contribution (MR) would become the following:

$$MR_i = \frac{d\sigma_P}{dw_i}$$

Also, the total marginal contribution would become the following:

$$TRC = w_i \frac{d\sigma_P}{dw_i}$$

Finally, the percentage total risk contribution (PTRC) is the following:

$$PTRC = \frac{TRC_i}{RM} = w_i \frac{d\sigma_P}{dw_i} \frac{1}{\sigma_P}$$

Now let's implement these measures into an example of a portfolio with a 2-asset class universe. We should know by now that the portfolio's standard deviation with 2 asset classes is the following:

$$\sigma_P = \sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A \sigma_B w_A w_B CORR_{A,B}}$$

The marginal risk contribution for asset class A in portfolio P is the following:

$$MR_{\sigma,A} = \frac{d\sigma_P}{dw_A}$$

$$= \frac{1}{2}\left[\left((\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A \sigma_B w_A w_B CORR_{A,B}\right)\right]^{-\frac{1}{2}} \left[2(\sigma_A w_A)\sigma_A + 2\sigma_A \sigma_B w_B CORR_{A,B}\right]$$

$$= \frac{2(\sigma_A w_A)\sigma_A + 2\sigma_A \sigma_B w_B CORR_{A,B}}{2\sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A \sigma_B w_A w_B CORR_{A,B}}}$$

$$= \frac{(\sigma_A^2 w_A) + 2\sigma_A \sigma_B w_B CORR_{A,B}}{\sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A \sigma_B w_A w_B CORR_{A,B}}}$$

Respectively, the marginal risk contribution for asset class B in portfolio P is the following:

$$MR_{\sigma,B} = \frac{d\sigma_P}{dw_B}$$

$$= \frac{1}{2}\left[\left((\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A \sigma_B w_A w_B CORR_{A,B}\right)\right]^{-\frac{1}{2}} \left[2(\sigma_B w_B)\sigma_B + 2\sigma_A \sigma_B w_A CORR_{A,B}\right]$$

$$= \frac{2(\sigma_B w_B)\sigma_B + 2\sigma_A\sigma_B w_A CORR_{A,B}}{2\sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A\sigma_B w_A w_B CORR_{A,B}}}$$

$$= \frac{(\sigma_B^2 w_B) + 2\sigma_A\sigma_B w_A CORR_{A,B}}{\sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A\sigma_B w_A w_B CORR_{A,B}}}$$

The total risk contribution for asset class A is the following:

$$TRC_A = w_A MR_A = w_A \frac{(\sigma_A^2 w_A) + 2\sigma_A\sigma_B w_B CORR_{A,B}}{\sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A\sigma_B w_A w_B CORR_{A,B}}}$$

Respectively, the total risk contribution for asset class B is the following:

$$TRC_B = w_B MR_B = w_B \frac{(\sigma_B^2 w_B) + 2\sigma_A\sigma_B w_A CORR_{A,B}}{\sqrt{(\sigma_A w_A)^2 + (\sigma_B w_B)^2 + 2\sigma_A\sigma_B w_A w_B CORR_{A,B}}}$$

It should be natural for us to say that the sum of these total risk contributions equals to the portfolio's risk. For our 2-asset class example, the sum is the following:

$$\sigma_P = TRC_A + TRC_B$$

We can generalise this concept considering n as the number of asset classes composing the portfolio P. We introduce the general formula for the portfolio's risk using the total risk contributions:

$$\sigma_P = \sum_{i=1}^{n} TRC_i$$

After having made a concrete application of these metrics on a small sized portfolio, we proceed with generalizing these two fundamental tools. We can generalize the marginal risk of asset class *i* in the following formula:

$$MR_{\sigma,i} = \frac{d\sigma_P}{w_i} = \frac{\sigma_i^{\,2} w_i + \sum_{i\neq j}^{n} \sigma_i \sigma_j w_j CORR_{i,j}}{\sigma_P}$$

$$= \frac{\sigma_i^{\,2} w_i + \sum_{i\neq j}^{n} COV_{i,j} w_j}{\sigma_P} = \frac{\sum_{j=1}^{n} w_j COV_{i,j}}{\sigma_P}$$

It important to point out that this formula applies for portfolio with more than 2 asset classes. We can say from this generalisation that the marginal risk of asset class *i* is influenced by the covariance between asset class *i* and *j* weighted to asset class j. By consequence, we know that any variation in these two components will lead to a change in the marginal risk metric. Let's assume that there has been an increase in the weight of asset *i* compared to a negative covariance between asset classes *i* and *j*: the marginal risk will be negative due to the non-positivity of the numerator. For the investor, it happens to be a positive aspect to consider since increasing the investment in asset class *i* will lead to a negative marginal risk which means that the portfolio's risk decreases about %MR if we add a unit of weight in asset class *i*. We can generalise this concept by saying that the numerator is the key element that impacts either positively or negatively on the portfolio's risk. We know that weights are positive due to the long only constraint therefore asset class's *j* weight cannot be negative. This implies that the only variable that could make MR either positive or negative is the covariance between these two asset classes. If this covariance is negative, then the marginal risk will work in the favour of the investor by decreasing the portfolio's risk with the increase of the weight of asset class *i*. This happens when the weights are consistent. If the covariance is positive, it means that the marginal risk is positive therefore, any increase in the weight of asset *i* will lead to an increase of %MR in the portfolio's risk which cannot be accepted by a rational investor.

After generalising the marginal risk, we proceed with generalising the total risk contribution metric as well:

$$TRC_{\sigma,i} = w_i \frac{d\sigma_P}{w_i} = w_i \frac{\sigma_i^2 w_i + \sum_{i \neq j}^{n} \sigma_i \sigma_j w_j CORR_{i,j}}{\sigma_P}$$

$$= \frac{w_i(\sigma_i^2 w_i + \sum_{i \neq j}^{n} COV_{i,j} w_j)}{\sigma_P} = \frac{w_i \sum_{j=1}^{n} w_j COV_{i,j}}{\sigma_P} = \frac{\sum_{j=1}^{n} w_i w_j COV_{i,j}}{\sigma_P}$$

We can also summarise the marginal risks in a Nx1 vector:

$$\nabla MR_\sigma = \frac{\Sigma w}{\sqrt{w' \Sigma w}}$$

To get the single marginal risk we use the following:

$$MR_{\sigma,i} = \frac{(\Sigma w)_i}{\sqrt{w' \Sigma w}}$$

$(\Sigma w)_i$ is the *ith* row of the column vector resulting from the product of the covariance matrix and the weights vector.

Instead, for the single total risk contribution we use this matrix formula:

$$TRC_{\sigma,i} = w_i \frac{(\Sigma w)_i}{\sqrt{w'\Sigma w}}$$

The generalised version of the portfolio's standard deviation through matrix algebra is the following:

$$\sigma_P = \sum_{i=1}^{n} TRC_{\sigma,i} = w' \frac{\Sigma w}{\sqrt{w'\Sigma w}} = w' \nabla MR_\sigma$$

These metrics are fundamental in terms of understanding the impact each quantitative method has in the risk dimension of a portfolio. We will be using these tools in the next paragraphs to understand the 2 main strategies of this thesis. For a concrete application, we could use the marginal risk to identify the asset class that has a relevant impact on the portfolio's risk. In this way, we could set a constraint for this specific asset class that has a high positive marginal risk therefore, we limit the weight of such asset class which could save us from an unwanted high exposure to the risk dimension.

We proceed with implementing these tools in a concrete example. Let's assume investor X has a portfolio P with the following annual returns (we have used this dataset already with Markowitz's model):

Months	Returns 1	Returns 2	Returns 3	Returns 4
1	5,89%	7,89%	-2,25%	2,20%
2	3,67%	5,42%	9,98%	-9,55%
3	2,60%	5,41%	4,16%	-5,05%
4	4,67%	3,90%	-2,98%	3,81%
5	1,85%	-2,16%	6,02%	21,35%
6	3,27%	4,10%	4,16%	6,39%
7	-0,79%	1,49%	2,57%	5,02%
8	0,83%	1,57%	3,08%	-2,30%
9	1,48%	-2,16%	5,13%	9,81%
10	0,71%	4,39%	-2,99%	3,56%
11	0,86%	2,30%	3,91%	3,78%
12	0,09%	1,40%	1,24%	2,18%

We should know by now how to calculate the asset class's average returns and their annual standard deviations. In the next table we summarize these asset class's volatilities:

Asset Class 1	Asset Class 2	Asset Class 3	Asset Class 4
1,98%	3,01%	3,90%	7,69%

Other than the returns and volatilities of these asset classes, we must include another important element which is the asset class's weight. We assume the following distribution of weights:

	Asset Class 1	Asset Class 2	Asset Class 3	Asset Class 4
W	34%	15%	10%	41%

We also include the covariance matrix that we will be using to calculate the risk distribution for this portfolio:

COV X/Y	Asset Class 1	Asset Class 2	Asset class 3	Asset Class 4
Asset Class 1	**0,000390559**	0,000363055	-0,000114367	-0,000261119
Asset Class 2	0,000363055	**0,000905681**	-0,000391391	-0,001562492
Asset Class 3	-0,000114367	-0,000391391	**0,001518809**	-0,00022248
Asset Class 4	-0,000261119	-0,001562492	-0,00022248	**0,005906055**

In the following table we have the marginal risks (MR) of all 4 asset classes:

	MR(i)	W(i)	Annual std dev
Asset Class 1	0,242%	34%	1,976%
Asset Class 2	-1,479%	15%	3,009%
Asset Class 3	-0,130%	10%	3,897%
Asset Class 4	7,305%	41%	7,685%

It is noticeable how this portfolio is characterized by 2 asset classes that have negative marginal risks and the remaining 2 having positive marginal risks. As explained earlier, having a negative marginal risk refers to a decrease of the portfolio's risk when there is an increase in the investment in asset class I, whereas a positive marginal risk implies an increase in the portfolio's risk when we increase the asset class's weight. It is natural for us to say that it is preferrable to have in investor X's portfolio relevant weight percentages in asset classes such as n° 2 and n° 3 since they will help with the reduction of the portfolio's risk.

We also include the total risk contribution (TRC) as well as the percentage of the total risk contribution (PTRC) in the next table:

	MR	TRC	PTRC
Asset Class 1	0,242%	0,082%	2,894%
Asset Class 2	-1,479%	-0,222%	-7,808%
Asset Class 3	-0,130%	-0,013%	-0,457%
Asset Class 4	7,305%	2,995%	105,371%

We stated earlier that asset classes 2 and 3 have negative marginal risks which is an aspect that a rational investor would want in its portfolio.

We proceeded with calculating the total risk contribution in which we simply multiply the marginal risk to the asset class's weight. We can see the marginal risk as the risk variation with only one increased unit while the total risk contribution as all the units of risk asset class i contributes to the portfolio risk. In fact, by summing all total risk contributions we get the portfolio's risk which is the case in out example. For the overall portfolio risk, which is about 2,842%, asset class 1 contributes about 0.082%. The asset class which has a relevant contribution to the overall risk is asset class 4 which composes the portfolio's risk about 2.995%. It might be questionable how this total risk contribution is higher than the portfolio's risk itself, however asset classes 2 and 3 contribute with decreasing slightly the total risk contribution of asset class 4. The portfolio's risk would have been higher if the total risk contributions of asset classes 2 and 3 were positive thus leading asset class's 4 total risk contribution being lower than the portfolio's risk. Asset class 4 has been contribution significantly in reducing the portfolio's risk about 0.222%.

We proceeded with calculating the percentage total risk contribution which calculates the amount in percentage of risk each asset class occupies in the total portfolio risk. Simply let's imagine a international class in which 20% of the students come from the US, another 20% of students are from Switzerland and the remaining 60% from Italy. We notice that the sum of these percentages is 100% which equals to the whole class. Now let's apply this same concept into the percentage total risk contribution. Our class becomes the portfolio's risk and the students become asset classes. Our portfolio's risk is composed by 2.884% of asset class 1 (which is the percentage total risk contribution of asset class1), -7.808% of asset class 2, -0.457% of asset class 3 and finally 105.371% of asset class 4. In this case as well, we might question why asset class 4's percentage is higher than 100% which is the portfolio's risk. Earlier we developed a concept which applies to this case as well: even if the PTRC of asset class 4 is higher than 100%, there are asset classes such as n° 2 and 3 that balance (reduce) the impact in the portfolio's risk. If there weren't negative marginal risks, therefore negative total risk contributions and percentage total risk contributions then asset class's 4 percentage total risk contribution will be lower than 100%. We can notice that asset class 4 is the main asset class that contributes to the overall risk of this portfolio, however asset class 2 contributes with the slight reduction of this risk.

Now that we introduced these decomposition metrics that help us dive deep into the impact each asset class's risk has into the portfolio risk, we proceed with introducing one of the two main quantitative methods of this thesis: the Global Minimum Variance strategy.

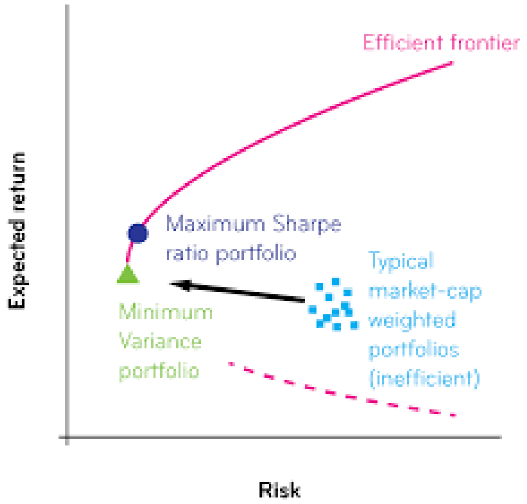
2.3 The Global Minimum Variance Portfolio GMVP

We introduced in the previous paragraphs risk decomposition metrics that, as we know by now, are necessary to understand the impact each asset class has in the portfolio's risk. We now present the first risk-based strategy which is one of the two main topics of this thesis: the Global Minimum Variance Optimisation.

We have seen this strategy indirectly already with the Mean-Variance Optimisation by Markowitz reason why its framework is quite like the Markowitz's one. This mean-free strategy suggests a portfolio located on the Efficient Frontier which, as we have seen, is a curve that has a set of optimal portfolios that have the highest return for a given standard deviation (risk) and vice versa the lowest risk (standard deviation) for a given return. We also mentioned in the previous chapter that portfolios that lie in this curve have the best trade-off between risk and return. Instead, those under this curve as sub-optimal portfolios since for a given risk level they have a much lower return than the ones in the curve itself. The same happens with portfolios on the right side of the curve: these are also sub-optimal portfolios in the sense that they have a much higher risk level for a given return.

It might seem contradicting the fact that a mean-free portfolio is linked with an Efficient Frontier that is based on the trade-off between the risk and return metrics. The Global Minimum Variance Portfolio is the one from which the Efficient Frontier starts. For this reason, it is in the left most end of the curve.

To make its location more understandable we introduce the following picture:



https://www.google.com/url?sa=i&url=https%3A%2F%2Fwww.stoxx.com%2Fdocument%2FOthers%2Fmarketing%2FSTOXX_MinVar_Paper.pdf&psig=AOvVaw2oeCtkHaLHell_iRQMoIN5&ust=1719130209788000&source=images&cd=vfe&opi=89978449&ved=0CA8QjRxqFwoTCIi_z6Ph7oYDFQAAAAAdAAAAABAE

It is noticeable that this portfolio is characterized by the lowest volatility, therefore it lies on the left end of the curve. After this introduction, we should dive into how we determine analytically this portfolio. In fact, we mentioned that its framework is like the Markowitz's optimisation framework. However, this optimizer does not have as its inputs the returns, risks and covariances; instead, it uses mainly the estimates of risks and covariances.

It should be intuitive to say that minimising portfolio variance means minimising the portfolio's standard deviation also known as the volatility. This method of minimising the volatility can be applied only to the Global Minimum Variance Portfolio therefore not on the other portfolios since they strongly depend on the estimated expected returns.

Mathematically speaking, we minimize the portfolio variance, and we limit the optimization using 2 constraints from the 3 that we have seen with Markowitz's optimization which are the budget constraint and the non-negativity constraint.

We reintroduce the formula that we need to optimize in this mean-free strategy:

$$\sigma_p = \sqrt{\sum_{i=1}^{n}(\sigma_i w_i)^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}} (w_i w_j \sigma_i \sigma_j CORR_{ij})}$$

For this strategy, we need to minimize the portfolio's volatility just as we have seen with the Mean-Variance Optimisation:

$$F(w) = Min\ \sigma_p = Min \sqrt{\sum_{i=1}^{n}(\sigma_i w_i)^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}} (w_i w_j \sigma_i \sigma_j CORR_{ij})}$$

The first constraint that we use to limit the minimizer is the budget constraint which implies that the sum of all weights equals to 1 (or 100%). We reintroduce the formula for this constraint:

$$\sum_{i=1}^{n} w_i = 1$$

The second and last constraint is the non-negativity constraint. It is fundamental adding it into the framework to avoid the speculative game played by some investors. We follow with its formula:

$$w_i \geq 0$$

The complete framework consists of putting these three components in a system:

$$
\begin{cases}
Min \sqrt{\sum_{i=1}^{n} (\sigma_i w_i)^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}} (w_i w_j \sigma_i \sigma_j CORR_{ij})} \\
\sum_{i=1}^{n} w_i = 1 \\
w_i \geq 0
\end{cases}
$$

In matrix connotation we simply have the following:

$$
\begin{cases}
Min\, w'\Sigma w \\
w'e = 1 \\
[w] \geq 0
\end{cases}
$$

We proceed with implementing this strategy in the following 12-year returns dataset:

A.C. 1	A.C. 2	A.C. 3	A.C. 4
5,89%	7,89%	-2,25%	2,20%
3,67%	5,42%	9,98%	-9,55%
2,60%	5,41%	4,16%	-5,05%
4,67%	3,90%	-2,98%	3,81%
1,85%	-2,16%	6,02%	21,35%
3,27%	4,10%	4,16%	6,39%
-0,79%	1,49%	2,57%	5,02%
0,83%	1,57%	3,08%	-2,30%
1,48%	-2,16%	5,13%	9,81%
0,71%	4,39%	-2,99%	3,56%
0,86%	2,30%	3,91%	3,78%
0,09%	1,40%	1,24%	2,18%

For these past returns we have the following estimated returns and standard deviations:

	Returns	Std Dev
Asset Class 1	2,10%	1,98%
Asset Class 2	2,80%	3,01%
Asset Class 3	2,67%	3,90%
Asset Class 4	3,43%	7,69%

We include both the covariance and correlation matrixes that are necessary for this framework:

COV X/Y	Asset Class 1	Asset Class 2	Asset class 3	Asset Class 4
Asset Class 1	**0,000390559**	0,000363055	-0,000114367	-0,000261119
Asset Class 2	0,000363055	**0,000905681**	-0,000391391	-0,001562492
Asset Class 3	-0,000114367	-0,000391391	**0,001518809**	-0,00022248
Asset Class 4	-0,000261119	-0,001562492	-0,00022248	**0,005906055**

CORR X/Y	Asset Class 1	Asset Class 2	Asset class 3	Asset Class 4
Asset Class 1	**1**	0,610438556	-0,148493089	-0,171927908
Asset Class 2	0,610438556	**1**	-0,333711436	-0,675587038
Asset Class 3	-0,148493089	-0,333711436	**1**	-0,074283378
Asset Class 4	-0,171927908	-0,675587038	-0,074283378	**1**

We proceed with implementing this framework into our example:

	Annual Return	Annual Std Dev	Weights
Asset Class 1	2,32%	2,10%	0,0%
Asset Class 2	3,19%	3,16%	53,7%
Asset Class 3	2,67%	4,08%	27,9%
Asset Class 4	3,43%	7,69%	18,4%
Portfolio	3,09%	0,838%	100%
Weighted average risk	4,25%		
Diversification benefit	80,28%		

This optimized portfolio has an annual return of 3.09% and volatility about 0.838%. We know by now that the Global Minimum Variance Optimization minimizes the portfolio's risk, and in our case 0.838% is the minimum risk level that this portfolio has. It is important to specify that the is no lower risk for this portfolio than the one we have just seen. With this framework, we have a diversification benefit of 80.28% meaning that this optimization saved us from a much higher risk level. It is noticeable how this quantitative strategy accumulates the investments into the asset classes with the lowest risk levels. In fact, asset class 2 has a 53.7% weight which is in our case the highest weight inn an asset class that almost has the lowest standard deviation. Another aspect to consider is the correlations of asset class 2: it noticeable how the framework concentrates the weight in the second asset class since its correlations are the highest negative ones among the rest. This extreme loading to one asset class is quite a general characteristic of the global minimum variance optimization. Another matter is the exclusion of asset class 1 from the investment universe, which goes against the diversification benefit. We have seen this issue with the Mean-Variance Model by Markowitz.

Now let's dive into this portfolio's risk decomposition:

	WEIGHTS	MR	TRC	TRC
Asset Class 1	0%	1,574%	0,000%	0,0%
Asset Class 2	54%	0,838%	0,450%	53,7%
Asset Class 3	28%	0,838%	0,234%	27,9%
Asset Class 4	18%	0,838%	0,154%	18,4%

Firstly, we can notice how all marginal risks are about 0.838%, except the one related to asset class 1 which has been excluded from the portfolio's universe**.** This is in fact one of the many characteristics of the global minimum variance optimization: the marginal risk of the asset class that has been excluded, therefore the one that has a 0% weight, is always different from the rest of asset classes that have been included in the investment universe.

Having marginal risks that are equal does not mean that the total risk contributions are equal as well. We know by now that we calculate the total risk contribution by multiplying the marginal risk to its weight, and we also know that the weights differ from each other meaning the total risk contribution will differ as well from each other. We can tell already that asset class 2 has the highest total risk contribution since it has the highest weight among all asset classes. In fact, its percentage total risk contribution is about 56.3% out of the total contribution of the portfolio which is 100%. We can also see it this way around: 0.45% is about 53.7% of the 1.13% of the portfolio's volatility.

Another important point to focus on is the increase of volatility the portfolio will face once asset classes have increased standard deviations or correlations. Let's imagine an increase in all 4 asset classes' standard deviations. The global minimum variance framework will heavily concentrate the weight into the asset class that has the lowest standard deviation and lowest correlations. However, seen how the standard deviations have increased, the optimiser still picks the asset classes with the lowest volatilities which in our case have increased. By consequence, the portfolio's volatility is the lowest due to the GMV framework, however it is remarked by an increase due to the increase in the asset classes' volatilities. We can conclude by saying that there is a direct relation among the portfolio's risk through the GMV framework and the individual volatilities or correlations of asset classes.

We can finally summarize the relevant components of this mean-free strategy in the following list:

GMVP Characteristics
• The portfolio coming from this framework is ideal for risk adverse investors. • The optimization's output is a portfolio having the lowest standard deviation/risk. • Partial inclusion of asset classes meaning some asset classes are excluded from the investment universe. • Marginal risks are equal except the ones that are associated to the asset classes that have been excluded • Increase in the individual volatilites or correlations will increase the portfolio's risk

2.4 The most Diversified Portfolio (MDP)

The Most Diversified Portfolio approach has been the most recent one. In fact, it has been introduced by Yves Choueifaty and Yves Coignard in 2006.

The aim of this strategy is to maximise the diversification in a portfolio. These two economists propose an asset allocation strategy that focuses on the long-only concept (meaning weights must be positive, therefore ≥0) and on achieving at the same time the highest diversification ratio. We find these two components in the so called Most Diversified Portfolio (MDP).

The diversification ratio of a portfolio (DR_P) is the ratio of the weighted average of asset classes' standard deviation to the volatility of the portfolio with the same asset classes and allocations. We introduce the formula for this ratio:

$$DR_P = \frac{\sum_{i=1}^{n} w_i \sigma_i}{\sigma_P}$$

In matrix notation we have the following:

$$DR_P = \frac{w'\sigma}{\sqrt{w'\Sigma w}} \text{ or } DR_P = \frac{w'\sqrt{diag(\Sigma)}}{\sqrt{w'\Sigma w}}$$

Where sigma is an N*1 vector of asset classes' volatilities and diag(Σ) indicates the vector N*1 of diagonal elements of the covariance matrix.

This metric helps us quantify how much the portfolio's risk would have been higher if all constituents were perfectly correlated. Let's say this diversification ratio is about 1.58: this means that if asset classes had CORR=1, the portfolio's risk would have been 58% higher than what we have now. We can also see it the other way around: it is as if the portfolio's diversification (CORR≠1) helped us reduce the portfolio's risk by 58%.

We should know by now that the numerator of the diversification ratio represents the highest standard deviation a portfolio could reach. This should remind us of a formula we have seen in chapter 1: the diversification benefit formula. We reintroduce this metric for a better understanding:

$$\frac{\sum_{i=1}^{n} \sigma_i w_i - \sigma_{REAL}}{\sum_{i=1}^{n} \sigma_i w_i} = x\%$$

This formula should be familiar to us in the way that the higher the diversification is among the constituents, the higher this diversification benefit is going to be, and by consequence, the lower the portfolio's standard deviation will be.

In fact, we can say that the diversification ratio provides an indirect measurement of the benefit of diversification in terms of risk saving. The aim has always been risk saving meaning taking the lowest risk given the selected universe of asset classes.

Now let's analyse the different scenarios we might stumble into while using this mean-free approach. It would be natural for us to say that the diversification ratio cannot be negative. In the numerator we have weights that are positive for the long-only constraint and the respective volatilities of asset classes that are always higher than 0, therefore positive. In the denominator we have the portfolio's risk which we know is positive. We can now exclude the assumption of the diversification ratio being negative.

The lowest value this ratio can achieve is 1. This happens when the numerator and the denominator are qual therefore when the weighted average risk is equal to the portfolio's risk. It means that the portfolio has reached the highest risk and on another note the lowest diversification. We can say that such portfolio is poorly diversified. The value of this ratio cannot be between 0 and 1 since the portfolio's risk will never be higher than the weighted average risk.

The higher the diversification ratio is the more diversified the portfolio will be. From one side, an investor would want the lowest portfolio's volatility which is what these mean-free approaches focus on. From the other side, we focus on maximising the diversification ratio. For example, by expanding our investment universe, we increase the weighted average risk and the same time we increase the gap between the weighted average risk and the portfolio's risk. This means that by increasing the investment universe, we will increase the risk saving as well by using the diversification concept. An increase in the weighted average risk can be translated into an increase in correlations due to the increase of asset classes, and an increase in correlations leads us to a more balanced portfolio and mostly a lower portfolio's risk.

We now proceed with the introduction of the framework for this mean-free strategy. The objective function is the diversification ratio that must be maximised. We introduce the formula:

$$MAX\left(\frac{\sum_{i=1}^{n} w_i\sigma_i}{\sqrt{\sigma^2{}_P}}\right)$$

In matrix notation it is the following:

$$MAX\left(\frac{w'\sqrt{diag(\Sigma)}}{\sqrt{w'\Sigma w}}\right)$$

Other than the objective function, there are 2 constraints that are needed for this framework. The first constraint is the long-only constraint which stands for the positivity of each asset class's weight. The second constraint is the budget constraint that relies on the sum of weights being equal to 1 which stands for the investor's budget.

The final framework is the following:

$$\begin{cases} MAX\left(\frac{w'\sqrt{diag(\Sigma)}}{\sqrt{w'\Sigma w}}\right) \\ w'e = 1 \\ [w] \geq 0 \end{cases}$$

We now proceed with implementing this strategy into the following 12-year returns dataset:

AC 1	AC 2	AC 3	AC 4
5,89%	7,89%	-2,25%	2,20%
3,67%	5,42%	9,98%	-9,55%
2,60%	5,41%	4,16%	-5,05%
4,67%	3,90%	-2,98%	3,81%
1,85%	-2,16%	6,02%	21,35%
3,27%	4,10%	4,16%	6,39%
-0,79%	1,49%	2,57%	5,02%
0,83%	1,57%	3,08%	-2,30%
1,48%	-2,16%	5,13%	9,81%
0,71%	4,39%	-2,99%	3,56%
0,86%	2,30%	3,91%	3,78%
0,09%	1,40%	1,24%	2,18%

In the following table we have the optimized Most Diversified Portfolio:

	Annual Return	Annual Std Dev	Weights
Asset Class 1	2,10%	1,98%	0%
Asset Class 2	2,80%	3,01%	59%
Asset Class 3	2,67%	3,90%	27%
Asset Class 4	3,43%	7,69%	21%
Portfolio	3,09%	1,22%	100%
Diversification Ratio	3,635		
Weighted average risk	4,44%		
Diversification benefit	72,49%		

We can see how the portfolio's risk is 1.22% which is the lowest compared to the highest diversification ratio this portfolio could reach. In fact, we have a 264% of risk being saved just from diversifying the portfolio, meaning this portfolio's risk would have been 264% higher than what we have now. The investor has a diversification benefit of 72.49% compared to a portfolio in which correlations are equal to 1. It is noticeable that asset class 1 has been excluded from the investment universe therefore we can say that the most diversified approach does not employ the entire investable set.

Now let's analyse the risk decomposition of our Most Diversified Portfolio:

	MR	TRC	PTRC
Asset Class 1	1,054%	0,000%	0,000%
Asset Class 2	0,828%	0,489%	40,017%
Asset Class 3	1,072%	0,289%	23,624%
Asset Class 4	2,114%	0,444%	36,359%

We mentioned earlier that asset class 1 has been excluded from the portfolio's universe. In fact, its total risk contribution is equal to 0%. Unlike the global minimum variance optimisation in which marginal risks are equal, we can notice how in the most diversified framework marginal risks differ from each other. Asset class 2 is the one that contributes most to the portfolio's risk. It composes about 40% of the portfolio's risk. We know already that the sum of the percentage total risk contributions is equal to 100% and the sum of the total risk contributions is equal to the portfolio's risk which is 1.22%.

We know that the portfolios with max diversification ratio are the most diversified ones. In fact, we can see this diversification concept as the maximisation of the Sharpe Ratio. The Sharpe Ratio is the quotient of the difference between the excess portfolio return and the standard deviation of the portfolio. We introduce its formula:

$$SR = \frac{\overline{R}_P - \overline{R}_{FREE}}{\sigma_P}$$

We can see the Sharpe Ratio as the positive or negative remuneration we have from the increase of the portfolio's volatility. A rational investor would want a positive remuneration therefore a SR>0. The Most Diversified approach gives higher long run returns at lower volatility levels. It is also much more efficient ex-post than the market benchmark, therefore $\overline{R}_P > \overline{R}_{FREE}$. If the portfolio's return increases while the portfolio's risk increases as well, then the most diversified portfolio is the portfolio that has the highest sharpe ratio on the efficient frontier. If the portfolio's return decreases with the increase of the portfolio's volatility, then this portfolio lies in the lower half of the efficient frontier. The diversification ratio of any long-only portfolio is always higher than 1 beside the mono-asset portfolios in which their DR is equal to 1. Maximising the diversification ratio can be seen as finding the highest coordinates of the objective function, therefore it can be seen as finding the tangency portfolio that has the highest value. For this reason, the most diversified portfolio is also called the tangency portfolio.

There has been a decomposition of the diversification ratio in two different components: the volatility-weighted average correlations and the concentration ratio.

The reformulation of the diversification ratio has been the following:

$$DR = \frac{1}{\sqrt{CORR_{MDP}(1 - CR_{MDP}) + CR_{MDP}}}$$

In this reformulation we have the $CORR_{MDP}$ which is the volatility-weighted average correlations. We introduce its formula:

$$CORR_{MDP} = \frac{\sum_{i \neq j}^{n} (w_i w_j \sigma_i \sigma_j) CORR_{ij}}{\sum_{i \neq j}^{n} (w_i w_j \sigma_i \sigma_j)}$$

If the correlation among asset class i and j decreases, it will reduce the numerator of the volatility-weighted average correlations. This means that this metric will be reduced due to the reduction of the correlation among these asset classes. However, we have seen that $CORR_{MDP}$ lies in the denominator of the diversification ratio which will make the diversification ratio increase due to the smaller denominator.

If $CORR_{ij}$ is equal to 1, it means that $CORR_{MDP}$ will also be equal to 1. This makes the portfolio lack diversification due to the high values in correlations among asset classes, therefore the diversification ratio reaches its lowest value which is 1.

The second metric is the concentration ratio, also called the volatility-weighted concentration ratio. We introduce its formula:

$$CR_{MDP} = \frac{\sum_{i=1}^{n} (w_i \sigma_i)^2}{(\sum_{i=1}^{n} w_i \sigma_i)^2}$$

We can notice how the concentration ratio does not take into consideration the correlations among asset classes. A mono-asset portfolio is characterized by a CR=1.

A decrease in this metric implies an increase in the diversification ratio. We can say that to increase the diversification ratio, we must minimize the concentration ratio and/or the volatility-weighted average correlations. Another important matter to mention is the fact that we only minimize the concentration ratio when all the correlations among asset classes are equal.

After analysing these two quantitative methos for the portfolio optimisation, we must apply these in a real dataset in the following chapter.

CHAPTER 3

Application of the two quantitative strategies in a real dataset

3.1 Introduction to the dataset

This thesis wouldn't be complete without a concrete application of the quantitative models into real data. For this reason, we selected an important dataset which covers the period from January 2001 till December 2023. As we can see this dataset covers a 22-year period. We are working on 11 asset classes which are the following: Euro Monetary; Euro Government Bond; US Government Bond; Euro Corporate Bond; Global Corporate Bond; Emerging Countries Bonds; High Yield Bonds; European Equity; US Equity; Pacific Equity and finally Emerging Countries Equity.

We can notice how this dataset is characterized by a wide diversification in the asset classes. In fact, we have the Euro Monetary which is characterized by liquidity, the lowest volatility and respectively the lowest return among the rest of the asset classes. Then we have the Bond group which usually lies between the Euro monetary and the equity group in terms of risks and returns. Finally, we have the equity group which is known for high volatilities and high returns in comparison to the rest of the investment universe. It is important to mention how we included not only the Euro zone, but also the global market including the US, the emerging countries and the Pacific area.

The investment in these asset classes have started the 29th of December 2000 in which it has been invested 100 euros in the Euro Monetary with the final amount being 138,56 euros. In the Euro Government Bond, it has been invested about 105.22 euro with the final amount of 206.25 euros. In the US Government Bond, it has been invested about 1157,97 euros with the final amount being 2058,11 euros. 111,86 euros have been invested in the Euro Corporate Bond having the final amount of 246,30 euros. Instead, in the Global Corporate Bond, we register an investment of 110,89 euros that made about 256,54 in December 2023. Then we have the investment in the Emerging Countries Bond which was about 206,74 euros that made later in December 2023 about 806,77 euros. Now we have the investment in the High Yield Bond which was about 330,55 euros that led to a final amount of 1375,63 euros. We have the European equity investment which was 132,79 and that made 309,44 euros. Then we have the US equity investment which was about 2899,02 euros that increase to 12191,55 euros. The last two investments are the Pacific Equity and the Emerging countries Equity respectively having invested in them about 112,17 euros and 106,44 euros and that increased into the following amounts: 258,34 euros and 482,49 euros.

We can notice that the investments that grew significantly are the investments made in the US Equity and the High Yield Bonds.

After introducing the dataset to the reader, we must introduce the respective returns as well as the standard deviations, correlations and covariances of these asset classes.

In the following table we introduce the returns and standard deviations:

ASSET CLASSES	Annual STD DEV	Annual RETURNS
Euro Monetary	0,51%	1,42%
Euro Government Bonds	3,83%	3,00%
US Government Bonds	9,72%	2,97%
Euro Corporate Bonds	4,25%	3,53%
Global Corporate Bonds	9,17%	4,07%
Emerging Countries Bonds	9,90%	6,42%
High Yield Bonds	10,01%	6,72%
European Equity	15,20%	4,85%
US Equity	15,14%	7,42%
Pacific Equity	14,33%	4,66%
Emerging Countries Equity	18,60%	8,34%

We introduce respectively the covariances and correlations among these asset classes:

Annual COV	EURO MONETARY	EURO GOVERN. BONDS	US GOVERN. BONDS	EURO CORP. BONDS	GLOBAL CORP. BONDS	EMERG. COUNTRIES BONDS	HIGH YIELD BONDS	EUROPEAN EQUITY	US EQUITY	PACIFIC EQUITY	EMERG. COUNTRIES EQUITY
EURO MONETARY	**0,000026**	0,000032	0,000012	0,000009	-0,000023	-0,000010	-0,000052	-0,000130	-0,000159	-0,000123	-0,000067
EURO GOVERN. BONDS	0,000032	**0,001470**	0,001155	0,001310	0,001324	0,001496	0,000798	0,000200	0,000414	0,000555	0,000302
US GOVERN. BONDS	0,000012	0,001155	**0,009444**	0,000075	0,007893	0,005610	0,003342	-0,003359	0,001597	0,000747	-0,002823
EURO CORP. BONDS	0,000009	0,001310	0,000075	**0,001808**	0,001231	0,001898	0,001917	0,002443	0,001994	0,002263	0,002834
GLOBAL CORP. BONDS	-0,000023	0,001324	0,007893	0,001231	**0,008410**	0,007279	0,006188	0,001330	0,005510	0,004484	0,002480
EMERG. COUNTRIES BONDS	-0,000010	0,001496	0,005610	0,001898	0,007279	**0,009805**	0,008559	0,005705	0,008332	0,007441	0,008728
HIGH YIELD BONDS	-0,000052	0,000798	0,003342	0,001917	0,006188	0,008559	**0,010029**	0,008606	0,010563	0,009230	0,011360
EUROPEAN EQUITY	-0,000130	0,000200	-0,003359	0,002443	0,001330	0,005705	0,008606	**0,023097**	0,018773	0,015932	0,021346
US EQUITY	-0,000159	0,000414	0,001597	0,001994	0,005510	0,008332	0,010563	0,018773	**0,022930**	0,015950	0,019038
PACIFIC EQUITY	-0,000123	0,000555	0,000747	0,002263	0,004484	0,007441	0,009230	0,015932	0,015950	**0,020543**	0,019777
EMERG. COUNTRIES EQUITY	-0,000067	0,000302	-0,002823	0,002834	0,002480	0,008728	0,011360	0,021346	0,019038	0,019777	**0,034585**

CORR	EURO MONETARY	EURO GOVERN. BONDS	US GOVERN. BONDS	EURO CORP. BONDS	GLOBAL CORP. BONDS	EMERG. COUNTRIES BONDS	HIGH YIELD BONDS	EUROPEAN EQUITY	US EQUITY	PACIFIC EQUITY	EMERG. COUNTRIES EQUITY
EURO MONETARY	**1,000**	0,162	0,025	0,042	-0,051	-0,020	-0,103	-0,169	-0,207	-0,169	-0,071
EURO GOVERN. BONDS	0,162	**1,000**	0,310	0,803	0,377	0,394	0,208	0,034	0,071	0,101	0,042
US GOVERN. BONDS	0,025	0,310	**1,000**	0,018	0,886	0,583	0,343	-0,227	0,109	0,054	-0,156
EURO CORP. BONDS	0,042	0,803	0,018	**1,000**	0,316	0,451	0,450	0,378	0,310	0,371	0,358
GLOBAL CORP. BONDS	-0,051	0,377	0,886	0,316	**1,000**	0,802	0,674	0,095	0,397	0,341	0,145
EMERG. COUNTRIES BONDS	-0,020	0,394	0,583	0,451	0,802	**1,000**	0,863	0,379	0,556	0,524	0,474
HIGH YIELD BONDS	-0,103	0,208	0,343	0,450	0,674	0,863	**1,000**	0,565	0,697	0,643	0,610
EUROPEAN EQUITY	-0,169	0,034	-0,227	0,378	0,095	0,379	0,565	**1,000**	0,816	0,731	0,755
US EQUITY	-0,207	0,071	0,109	0,310	0,397	0,556	0,697	0,816	**1,000**	0,735	0,676
PACIFIC EQUITY	-0,169	0,101	0,054	0,371	0,341	0,524	0,643	0,731	0,735	**1,000**	0,742
EMERG. COUNTRIES EQUITY	-0,071	0,042	-0,156	0,358	0,145	0,474	0,610	0,755	0,676	0,742	**1,000**

3.2 Application of the Global Minimum-Variance strategy

We have seen in the previous chapter the framework for this mean-free approach which mainly focuses on minimising the objective function. There are also two necessary constraints for this framework: the long-only constraint and the budget constraint. We proceed with implementing this approach to our dataset and we will analyse the risk distribution of the portfolio we get from the optimizer.

In the following table we have the global minimum variance portfolio:

	STD DEV	RETURNS	WEIGHTS	WEIGHTS
Euro Monetary	0,51%	1,42%	99,11911%	99,12%
Euro Government Bonds	3,83%	3,00%	0,00000%	0,00%
US Government Bonds	9,72%	2,97%	0,00000%	0,00%
Euro Corporate Bonds	4,25%	3,53%	0,00000%	0,00%
Global Corporate Bonds	9,17%	4,07%	0,03466%	0,03%
Emerging Countries Bonds	9,90%	6,42%	0,00000%	0,00%
High Yield Bonds	10,01%	6,72%	0,00000%	0,00%
European Equity	15,20%	4,85%	0,00022%	0,00%
US Equity	15,14%	7,42%	0,64066%	0,64%
Pacific Equity	14,33%	4,66%	0,20535%	0,21%
Emerging Countries Equity	18,60%	8,34%	0,00000%	0,00%
PORTFOLIO	**0,49%**	**1,46%**	**100,00%**	**100,00%**
Diversification benefit	22.20%			
Weighted average risk	0.63%			

The lowest standard deviation that this dataset can reach is about 0.49% considering that the framework tends to accumulate the investment into the asset class with the lowest volatility and lowest correlations. In this case the asset class that has these characteristics is asset class Euro Monetary which composes about 99.12% of the global minimum variance portfolio's budget. The portfolio's return is about 1.46% which is slightly higher than the return of asset class Euro Monetary. It is noticeable as well that the framework attributes small weight percentages to other 3 asset classes. We also have a diversification benefit of 22.20% which is the risk saved by the global minimum variance portfolio.

We proceed with analysing the risk distribution of the GMVP:

ASSET CLASS	MR	TRC	PTRC
Euro Monetary	0,49%	0,487%	99,12%
Euro Government Bonds	0,72%	0,0000%	0,00%
US Government Bonds	0,54%	0,0000%	0,00%
Euro Corporate Bonds	0,54%	0,0000%	0,00%
Global Corporate Bonds	0,49%	0,0002%	0,03%
Emerging Countries Bonds	1,24%	0,0000%	0,00%
High Yield Bonds	0,75%	0,0000%	0,00%
European Equity	0,49%	0,0000%	0,00%
US Equity	0,49%	0,0032%	0,64%
Pacific Equity	0,49%	0,0010%	0,21%
Emerging Countries Equity	1,97%	0,0000%	0,00%

Having calculated the marginal risks of the 11 asset classes, we proceeded with calculating the total risk contribution and the percentage total risk contribution. We must have expected a high total risk contribution in asset class Euro Monetary due to the significant weight attribution this asset class has. We can say that most asset classes have been excluded from the investment universe, and it is for this reason that most TRC are equal to 0. In some cases, we have a 0% TRC even when MR is equal to 0.49%. This means that the asset class has been included in the investment universe, therefore its weight is higher than 0. In this specific case, we have a very low weight percentage, reason why we have very small TRC values that are approximated to 0%. We saw in the previous chapter that one of the global minimum variance portfolio's characteristics is the fact that marginal risks in the included asset classes are equal. In this portfolio the MR for the asset classes included in the investment universe is about 0.49%. We verified that the sum of all TRC is equal to the portfolio's volatility. We conclude by saying that asset class Euro Monetary is the one that impacts mostly the portfolio's risk. However, we must say that there are other asset classes with higher marginal risks which have a higher impact on the risk dimension of our portfolio and luckily, due to the framework, these haven't been included in the investment universe of the global minimum variance portfolio.

The global minimum variance portfolio can be suggested to risk adverse investors who do not want any unwanted risk added in their portfolio even if the reward (which is the return) is much higher than the one that comes from this optimisation.

In the next paragraphs we apply the most diversified framework to our dataset.

3.3 Application of the Most Diversified strategy

After implementing the global minimum variance approach, we proceed with the most diversified strategy. This strategy focuses on maximising the objective function which, in this case, is the diversification ratio. We have seen that the diversification ratio gives as an output a positive value mostly equal or higher than 1. When the DR is equal to 1, it means that there is a poor diversification in the portfolio. A rational investor would want the maximum DR compared to the lowest risk.

We proceed with implementing this strategy in our dataset:

	STD DEV	RETURNS	WEIGHTS
Euro Monetary	0,51%	1,42%	87,44%
Euro Government Bonds	3,83%	3,00%	2,29%
US Government Bonds	9,72%	2,97%	4,82%
Euro Corporate Bonds	4,25%	3,53%	1.85%
Global Corporate Bonds	9,17%	4,07%	0,02%
Emerging Countries Bonds	9,90%	6,42%	0,00%
High Yield Bonds	10,01%	6,72%	0,00%
European Equity	15,20%	4,85%	2,90%
US Equity	15,14%	7,42%	0,00%
Pacific Equity	14,33%	4,66%	0,00%
Emerging Countries Equity	18,60%	8,34%	0,69%
PORTFOLIO	**0,80%**	**1,72%**	**100,00%**
Diversification Ratio (DR)	2.06		
Diversification benefit	51,42%		
Weighted average risk	1,63%		

With the Most Diversified approach, we have a portfolio with a return of 1.72% and a standard deviation of 0.80%. With the diversification ratio we managed to maximise the diversification for this portfolio while minimising at the same time its volatility. With this strategy, the portfolio's risk is about 106% lower than the actual weighted average risk. We also notice how we have more inclusion of asset classes in the investment universe than the one we saw in the global minimum-variance approach. We can also notice that the diversification benefit has a 51.42% in value, meaning that we saved about 51.42% of risk from the maximum risk percentage that this portfolio could have reached. Now we have 7 out of 11 asset classes included, however we can still notice the significant attribution of weight percentage in asset class US Government Bonds.

We proceed with analysing the risk decomposition in the most diversified portfolio:

ASSET CLASSES	MR	TRC	PTRC
Euro Monetary	0,25%	0,22%	26,86%
Euro Government Bonds	1,86%	0,04%	5,32%
US Government Bonds	4,72%	0,23%	28,39%
Euro Corporate Bonds	2,07%	0,04%	4,76%
Global Corporate Bonds	5,87%	0,00%	0,13%
Emerging Countries Bonds	6,97%	0,00%	0,00%
High Yield Bonds	6,22%	0,00%	0,00%
European Equity	7,38%	0,21%	26,70%
US Equity	8,26%	0,00%	0,00%
Pacific Equity	7,28%	0,00%	0,00%
Emerging Countries Equity	9,03%	0,06%	7,84%

The asset class that mostly contributes to the risk dimension of the most diversified portfolio is asset class US Government Bonds with a 28.39% contribution to the overall portfolio's standard deviation. As mentioned earlier, more asset classes have been included in the investment universe. For this reason, more asset classes are now contributing as well to the risk composition. Just after the US Government Bonds, we have a significant weight attribution in asset class Euro Monetary which contributes to the portfolio's risk about 26.86%. Lastly, the third asset class with the highest risk contribution is asset class European Equity. We can notice how asset class European Equity has a risk contribution close to the Euro Monetary one even if the weight attribution is much different between the two. European Equity has weight of just 2.90% in this portfolio compared to the 87.44% of the Euro Monetary one. This aspect is caused by the two different factors that make 24.62% close to 27.81%. From one side, asset class Euro Monetary has such important PTRC since it has a high weight percentage. On the other side, asset class European Equity has a high PTRC almost close to the Euro Monetary one since it has a high marginal risk. We can notice the wide gap between the two marginal risks which are about 0.25% for Euro Monetary and 7.38% for European Equity.

We simulated the returns of the most diversified portfolio, therefore having 276 returns in total. We wanted to know whether there were some sort of similar correlations between the portfolio's returns and the asset classes' returns. We introduce the formula:

$$CORR_{MDP,i}(R_{MDP};R_i)$$

For example, we calculated a correlation between asset class Euro Monetary and the MDP returns, and we got a value of 0.485. Then we proceeded with calculating the remaining of correlations with the remaining asset classes. We noticed that the correlations between the portfolio's returns and the returns of asset classes included in the most diversified portfolio have the lowest correlations which are around $\cong 0.485$. The only correlation that seems to be different from the rest of the correlations of asset classes included is the one between the MDP returns and asset class Global Corporate Bonds. In this case, the correlation was about 0.64 which was not among the lowest correlations. However, the weight attribution was very minimal which was, in fact, 0.02%. The remaining correlations of asset classes not included in the investment universe were much higher than the ones included. In fact, these correlations range from 0.507 to 0.703. We can say that there might be a link between the framework and these correlations as if the framework picked the asset classes with the lowest correlations with the returns of the most diversified portfolio.

The most diversified portfolio can be defined as a portfolio with the lowest risk compared to the highest diversification ratio. Such portfolio would not be suitable for a risk adverse investor since the risk reduction is in relative terms and not in absolute terms like in the global minimum variance framework. We conclude by saying that this portfolio is suggested to an investor who is aiming to maximise the diversification benefit even if it costs the investor a much higher risk than the one in the global minimum variance portfolio.

3.4 Conclusions

We began our journey with introducing the mean-variance optimisation by Markowitz. We mentioned that the framework was of high level. However, the inputs that are inserted into this framework are not of high quality, therefore it leads to a portfolio that has values that are not the real ones due to the estimation errors. We have also seen that estimation errors in expected returns are the highest ones, even higher than the risk ones and the correlation/covariance ones. To be able to build a real portfolio, we needed approaches that had to be much more accurate in the quality of their inputs. For this reason, we had to leave the mean-variance model behind and focus on two other quantitative methods: the global minimum variance approach and the most diversified approach.

These two methods solve the issue of estimation errors in expected returns by not considering these in their framework. In fact, they are called mean-free strategies since they do not consider returns in their optimisation. We learned how to set the framework and we applied them to a real dataset. We can say that the global minimum variance is a portfolio with the lowest volatility, and it lies in the most far left point of the efficient frontier. In fact, this portfolio is suitable for a risk adverse investor who is not willing to take more risk for a higher given return. The other portfolio we have seen is the most diversified portfolio which focuses on minimising the standard deviation while maximising the diversification ratio. This means that this portfolio has a much more inclusion of asset classes in the investment universe than the one in the global minimum variance portfolio. We conclude by saying that these approaches still focus on minimising the volatility without using the return dimension. However, we cannot say that they can be suggested to the same investor.

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